AD-A096 508 AUBURN UNIV AL DEPT OF MECHANICAL ENGINEERING F/G 12/1
A STOCHASTIC MODEL OF RELIABILITY GROWTH.(U)
FEB 80 D D PENROD DAAK40-79-M-0103
UNCLASSIFIED DRSMI/QR-80-1 NL

1 OF 1
AD
A096 508

END
DATE
FILMED
4-81
DTIC

(12) LEVEL II

AD A096508

TECHNICAL REPORT QR-80-1

A STOCHASTIC MODEL OF RELIABILITY GROWTH

BY

D. D. PENROD

FEBRUARY 18, 1980

U.S. ARMY MISSILE COMMAND
Redstone Arsenal, Alabama 35809

AD-B053707

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

DISPOSITION INSTRUCTIONS

DESTROY THIS REPORT WHEN IT IS NO LONGER NEEDED DO NOT RETURN IT TO THE ORIGINATOR

DISCLAIMER

THE FINDINGS IN THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION UNLESS SO DESIGNATED BY OTHER AUTHORIZED DOCUMENTS

TRADE NAMES

USE OF TRADE NAMES OR MANUFACTURERS IN THIS REPORT DOES NOT CONSTITUTE AN OFFICIAL INDORSEMENT OR APPROVAL OF THE USE OF SUCH COMMERCIAL HARDWARE OR SOFTWARE

18 DRXD 19 QR

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER QR-80-1	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A STOCHASTIC MODEL OF RELIABILITY GROWTH		5. TYPE OF REPORT & PERIOD COVERED FINAL TECHNICAL REPORT 3/21/79 - 12/31/79
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) DARRELL D. PENROD		8. CONTRACT OR GRANT NUMBER(s) DAAK40-79-M-0103
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Mechanical Engineering Auburn University Auburn, AL 36830		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element 644610 DD Form 1498
11. CONTROLLING OFFICE NAME AND ADDRESS Commander US Army Missile Command ATTN: DRSMI-QR Redstone Arsenal, AL 35898		12. REPORT DATE 18 February 1980
		13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 37		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release; Distribution Unlimited. 9 Final rept. 21 Mar - 31 Dec 79		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Reliability Growth, Reliability, Bayesian, Markov Chain		

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A stochastic reliability growth model is developed for one-shot devices which assumes reliability improvement occurs only after failures occur and the amount of improvement is proportional to the unreliability. The basic model is given by $P_{k+1}=P_k+C(1-P_k)$ where P_k is the reliability at the kth failure and C is a learning constant. Given an initial reliability P_o and the learning constant C, the expected reliability after n trials is calculated by Markov chain methods. Combining a discrete prior distribution of the initial reliability and learning

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

046244

Item #20 continued:
constant with test data using Bayes Theorem, a posterior distribution is developed which then is used to calculate the expected reliability growth curve. The model was applied to test data from three missile development programs. The results compared favorably with the AMSAA model currently in use and showed more logical initial growth. The model does not appear to be unusually sensitive to priors or size of input matrices.

Accession For		
NTIS GRA&I		✓
DTIC TAB		☐
Unannounced		☐
Justification		
By		
Distribution/		
Availability Codes		
Dist	Avail and/or Special	
A		

A STOCHASTIC MODEL OF RELIABILITY GROWTH

by

D. D. Penrod
Department of Mechanical Engineering
Auburn University
Auburn, Alabama 36830

February 18, 1980

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Introduction

Reliability growth prediction and assessment are widely used in development programs for complex systems. A number of mathematical models have been developed for describing the change in the system reliability as the design matures. Usually the growth results from a test, analysis and fix program; and most of the models have focused on repairable systems. While these methods may be applicable to ideal development programs (those with no human judgment errors), they are frequently lacking when applied to real programs. Perhaps part of this discrepancy is due the fact that human judgment as to what to fix and when can be a significant part of the reliability growth process. Moreover, there is often a learning process on the part of the designers as the test results unfold. It is this human intervention in the process that motivated the approach that is taken herein, namely a learning model approach. Stochastic learning models have been used for some time, and one recommended by Bush and Mosteller [1] has shown wide applicability. A reliability growth model based upon this method has some advantages over current methods, some of which are explored herein.

Stochastic Learning Model

The basic model under consideration applies to a sequence of experiments, each of which can be scored a success or failure. It is assumed that the system under consideration is only altered after failures occur. Therefore, the probability of success of a given trial depends only on the number of failures preceding that trial. Let P_k be the probability of success after k failures (and fixes) have occurred. The basic model is given by

$$P_{k+1} = P_k + C(1-P_k) \tag{1}$$

If $\pi(k,n)$ is defined to be the probability that exactly k failures occur in n trials, then the expected value of P after n trials, which is the reliability $R(n)$, is given by

$$R(n) = \sum_{0}^{n} P_k \pi(k,n) \tag{2}$$

Now for any given value of C and the initial probability P_o, the values of P_k can be calculated. Also, $\pi(k,n)$ can be calculated using the relationship

$$\pi(k,n+1) = P_k \pi(k,n) + (1-P_{k-1})\pi(k-1,n) \tag{3}$$

Finally, $R(n)$ is derived from $\pi(k,n)$ and P_k. Thus the entire process of generating growth curves $(R(n)$ versus $n)$ requires only the values of P_o and C.

In an actual development program, P_o and C are generally not known exactly. However, based on experience or preliminary analyses, they can be estimated. This suggests

a Bayesian approach where prior probability distributions are generated for P_o and C and the posterior distributions of these parameters are computed as test results become available.

A discrete probability distribution will be used to represent the prior and posterior probabilities. As will be seen, it will be convenient to introduce a joint distribution for P_o and C. Let

$$M_{ij} = \text{Probability}(P_o = p_i \text{ and } C = c_j) \tag{4}$$

For a given sequence S of test results, the posterior distribution

$$M'_{ij} = P(P_o = p_i, C = c_j | S)$$

is calculated according to

$$P(P_o = p_i, C = c_j | S) = \frac{P(P_o = p_i, C = c_j, S)}{P(S)}$$

$$M'_{ij} = \frac{P(S | P_o = p_i, C = c_j) M_{ij}}{\sum_{ij}\sum P(S | P_o = p_i, C = c_j) M_{ij}} \tag{5}$$

Now the entire process of growth curve generation is as follows:

1. Define a prior distribution for P_o and C.
2. Using test results, calculate the posterior distribution for P_o and C.
3. Use equations 1, 2, 3 to determine $\pi(k,n | P_o = p_i, C = c_j)$ and $R(n | P_o = p_i, C = c_j)$.
4. Take the expected value of $R(n)$ over the posterior distribution which is the expected reliability.

As can be seen, the process requires only an initial distri-

bution of P_o and C as input. The process is readily programmed on a digital computer. This was done on a PDP 11-70 for the purposes of this study. For the examples chosen, the required calculations were accomplished in several minutes, depending upon the number of trials and the number of points in the prior distribution.

Results

Using a 25 point prior distribution (five values of P_o and five values of C), growth curves were generated for three test cases. These are compared with those generated by the AMSAA model. The test data and assumed priors are given below. Calculated values are found in Appendix B.

SYSTEM #1

TOTAL NO. OF TESTS: 279

FAILURES OCCURRED AT TESTS: 1, 3, 4, 5, 6, 7, 9, 11, 13, 14, 19, 22, 24, 25, 26, 27, 28, 31, 36, 41, 44, 45, 47, 49, 53, 66, 69, 70, 71, 73, 75, 76, 77, 87, 91, 92, 100, 102, 108, 115, 116, 123, 124, 125, 129, 134, 135, 139, 143, 149, 151, 152, 155, 156, 157, 161, 166, 173, 179, 181, 182, 185, 194, 202, 207, 209, 229,

The prior distribution for System #1 is given as follows:

C \ P_o	.25	.40	.55	.70	.85
0	.04	.04	.04	.04	.04
.02	.04	.04	.04	.04	.04
.04	.04	.04	.04	.04	.04
.06	.04	.04	.04	.04	.04
.08	.04	.04	.04	.04	.04

SYSTEM #2

TOTAL NO. OF TESTS: 109

FAILURES OCCURRED AT TESTS: 3, 4, 5, 22, 26, 27, 30, 32, 37, 97, 103

SYSTEM #3

TOTAL NO. OF TESTS: 123

FAILURES OCCURRED AT TESTS: 2, 3, 4, 6, 8, 10, 11, 20, 24, 29, 34, 35, 46, 56, 61, 118

The prior distributions for systems #2 and #3 were the same

$C \backslash P_O$	.25	.40	.55	.70	.85
0	.04	.04	.04	.04	.04
.06	.04	.04	.04	.04	.04
.12	.04	.04	.04	.04	.04
.18	.04	.04	.04	.04	.04
.24	.04	.04	.04	.04	.04

Figure 1

SYSTEM #1 GROWTH CURVE

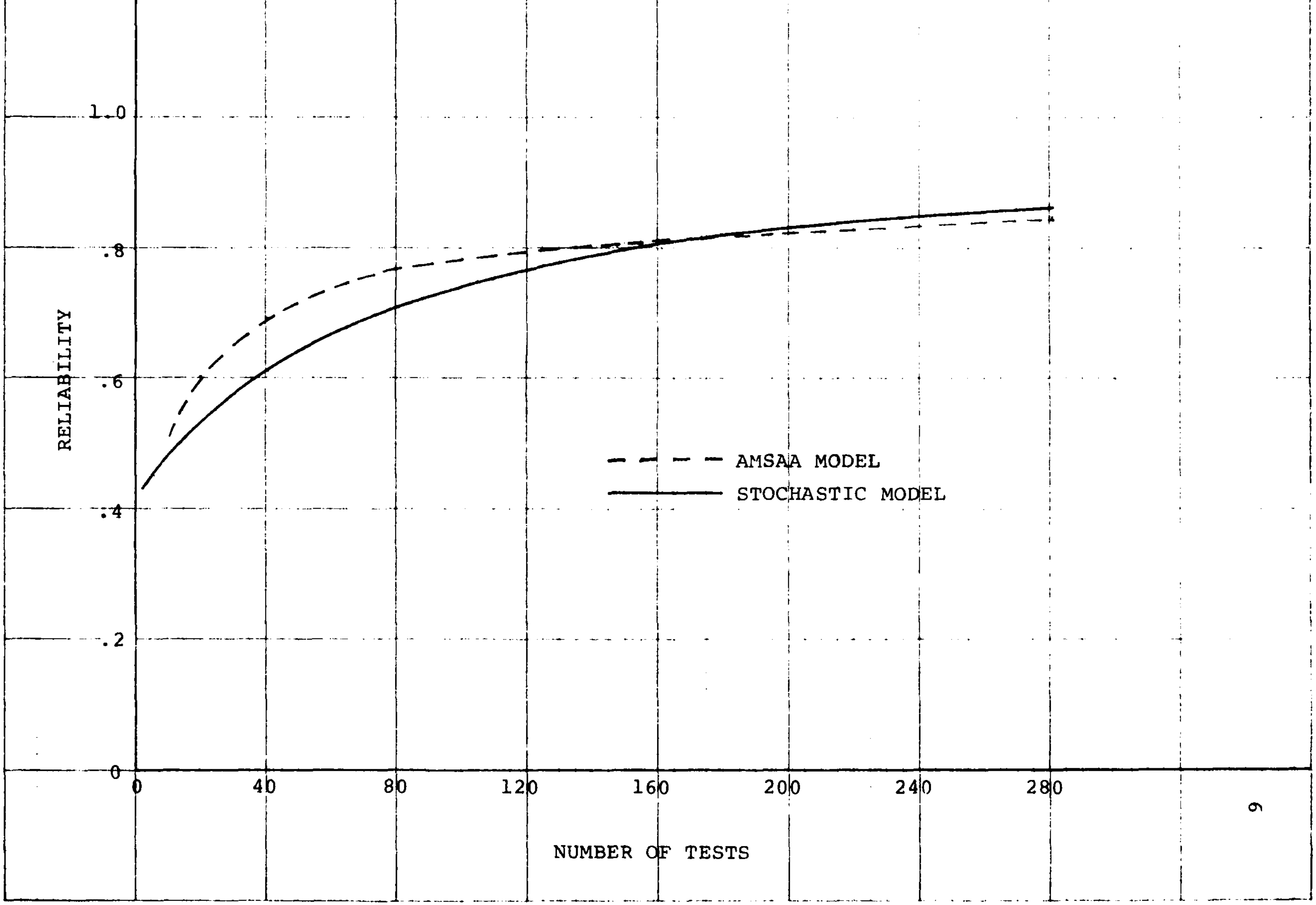


Figure 2

SYSTEM #2 GROWTH CURVE

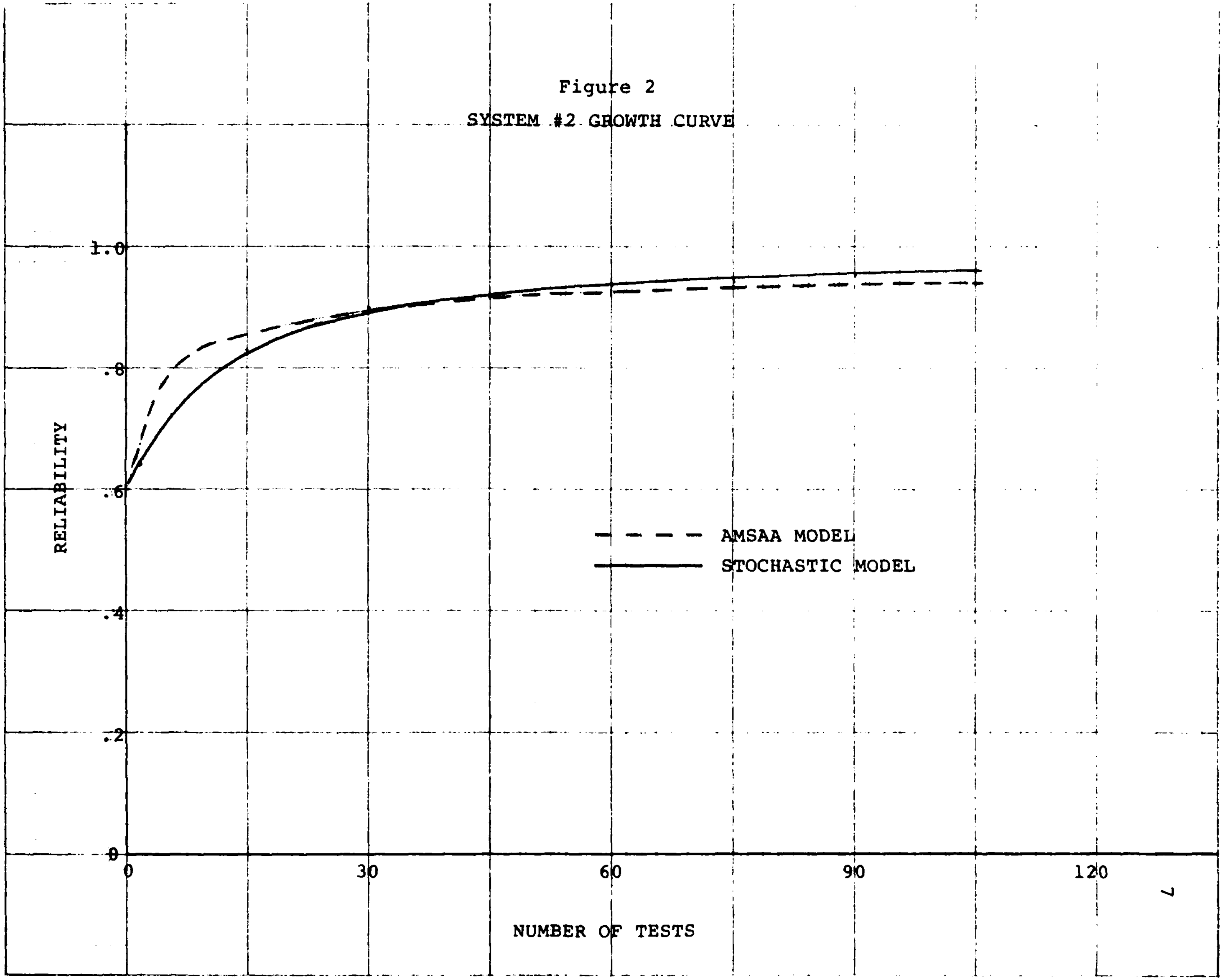
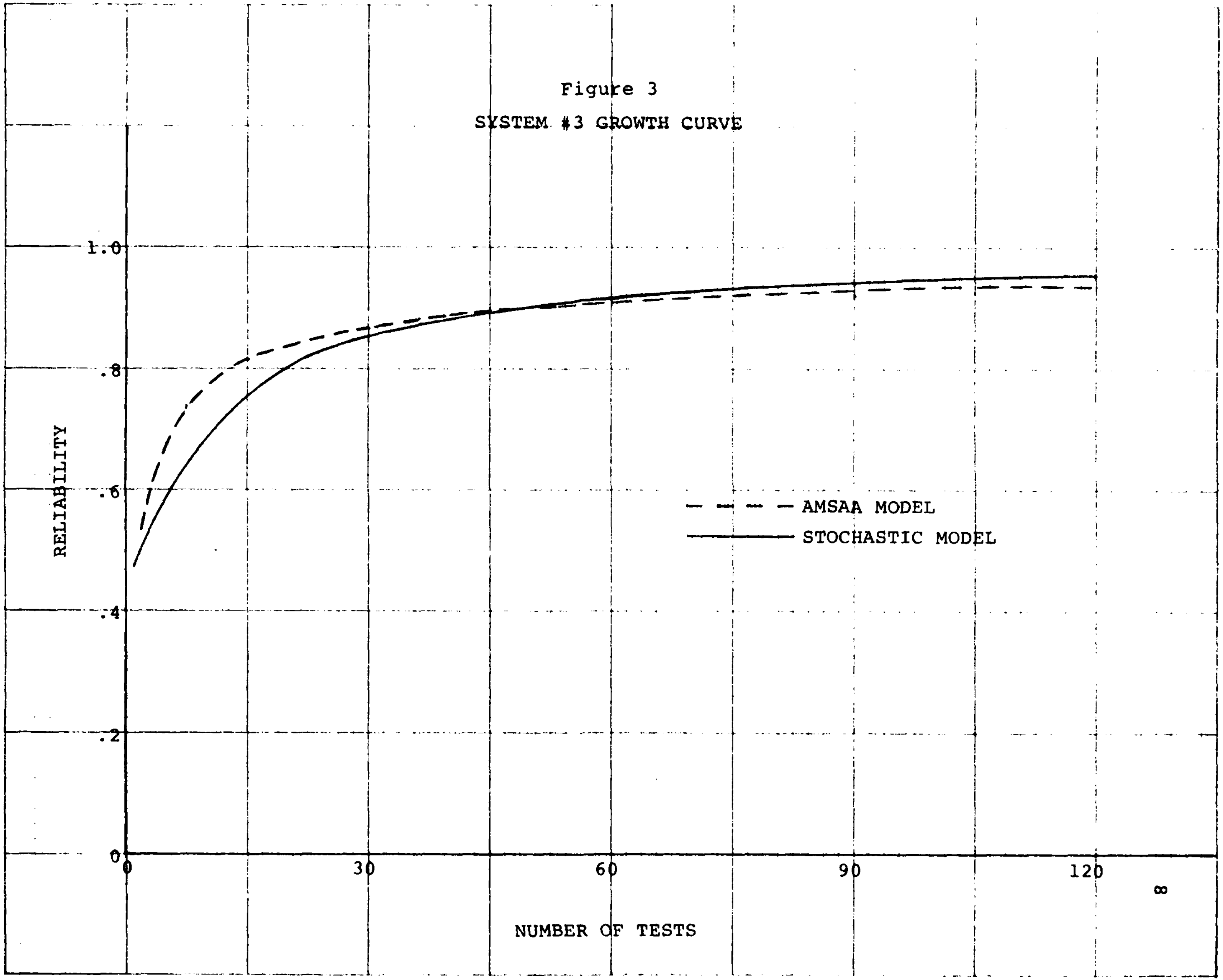


Figure 3

SYSTEM #3 GROWTH CURVE



In each of these cases, the AMSAA model has a steeper initial rise and a smaller slope thereafter. This type of behavior was anticipated and is one of the reasons for seeking other methods. The AMSAA curve, given by

$$1 - \lambda\beta T^{\beta-1}$$

will have an infinite slope at $T = 0$ if $\beta < 2$ (which is typical). Similarly, if growth curves are calculated on the basis of a small number of trials, the AMSAA model can yield results which are without meaning. In short, the curve for small values of T is of little value. Alternatively, the stochastic model uses the initial reliability of a design P_o as one of its parameters. This should yield better results for a small number of trials. Based on the three cases considered here, it would appear that the stochastic model gives a better representation of the reliability in the early phase of development and yields an ultimate value of the reliability which is in substantial agreement with the AMSAA method. This fact, plus the additional utility of the probability of program success capability (to be discussed in the next section) are sufficient to recommend the stochastic model. The additional effort required is small once the computer programs are written.

Methodology

The examples considered were already completed programs. The number of tests and ultimate reliability were specified. In a new program, these would be undetermined.

Also unknown would be the prior distribution for P_o and C. During the proposal phase of a system development, a preliminary design will presumably be proposed. Based on parts count, failure mode analysis, or experience with like designs, an approximate mean value $\overline{P}$ of P_o is chosen. Now let N be the total number of tests in the planned program, and P_u the ultimate reliability to be achieved by the end of the program.

The number of failures (and fixes) required to reach the desired reliability is

$$k = \frac{1}{\ell n\,(1-C)}\ \ell n \frac{1-P_u}{1-P_o} \tag{6}$$

and the expected number of tests to produce k failures is

$$N = \sum_{i=o}^{k-1} \frac{1}{(1-C)^i (1-P_o)} \tag{7}$$

These two equations can be solved simultaneously to find k and C.

As an example, consider the case N = 100, P_o = .5, P_u = .9. From Equation 6,

$$k = \frac{1}{\ell n\,(1-C)}\ \ell n \left(\frac{1-P_u}{1-P_o}\right)$$

$$k \doteq -\frac{1}{C}\,\ell n\,.2 = \frac{1}{C}\,\ell n 5$$

Equation 7 yields

$$100 = 2 \sum_{i=o}^{k-1} \frac{1}{\left(1-\frac{\ell n 5}{k}\right)^i}$$

This is satisfied, approximately, by k = 20. Thus

$$C \doteq \frac{1}{20} \ln 5 \doteq .08.$$

At this point, approximate mean values have been calculated for P_o and C. It remains only to select the complete joint distribution. Uniform or triangular distributions about P_o and C can be used to generate the priors M_{ij}. The sensitivity of the model does not appear to be too great so long as the mean values are the same. So the process of generating priors goes quite nicely with the process of initial reliability prediction and test program length. Since most, if not all of these steps, are required in the original proposal, it is a small task to set up the model originally for monitoring progress.

Probability of Success

A potentially useful product of the analysis described herein is the probability of reaching a specified ultimate reliability P_u. Denoted by P_o, it is calculated as follows:

$$P_s(n) = \sum_{P_k > P_u} \pi(k,n \mid P_o = p_i,\ C = c_j) M'_{ij}$$

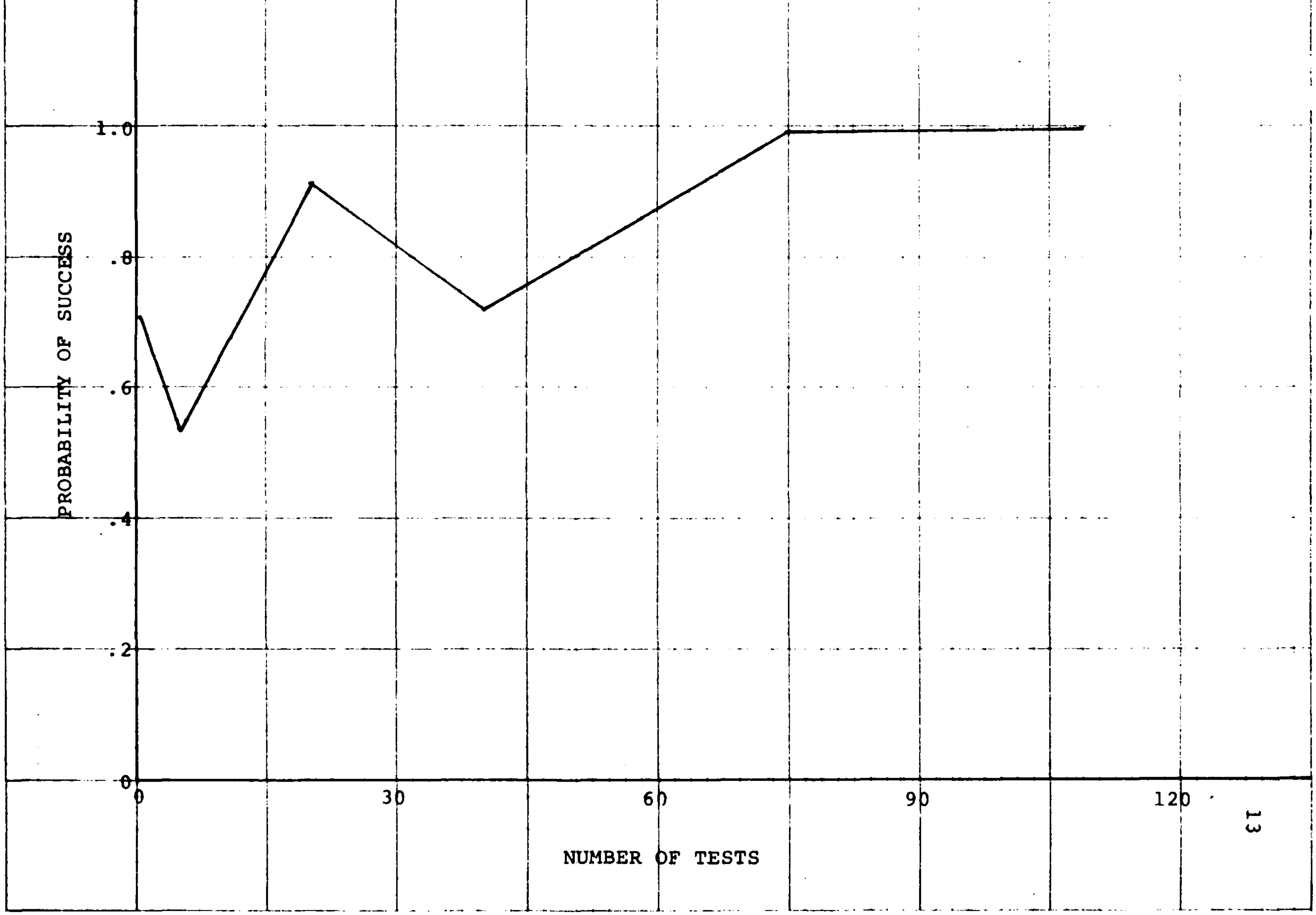
The summation is carried out over all i,j,k for which $P_k > P_u$. $P_s(n)$ is dependent upon n and the test results (through M'_{ij}). $P_s(n)$ for System #2 is plotted in the next graph the priors and test results are as before. Notice that estimated probability of reaching $P_u = .9$ was above .7 initially. After failures on tests 3,4,5; P_s had dropped to .53. There are two possible ways that such a diagram could be used:

1. As a trend indicator denoting either rapid or gradual changes in P_s.
2. In the manner of a control chart with lower limits being set on P_s at which point serious program review would be triggered.

Sensitivity

The matter of the sensitivity of the results to changes in the priors is of great interest. If dramatic changes in predicted reliability occur with small changes in the priors, this would mean that resulting growth curves would be heavily dependent upon the input distributions.

Figure 4
PROGRAM SUCCESS CURVE



Since these are judgmental, high sensitivity would call the entire method into question. Naturally, growth curves based on a few tests will be heavily influenced by changes in M_{ij}. This is illustrated by the accompanying curves which are growth curves based on one test, for three different priors. The base case is the same as was used previously for System #2. Variation #1 uses a distribution which weighs higher values of P_o more. Variation #2 weighs higher values of C more. Figure 5 gives the curves after one test. Figure 6 shows the resulting curves based on posteriors after 109 tests. Behavior typical of Bayesian models is seen, namely, the results are less dependent on priors as more information is included in the posteriors.

Sensitivity of a second type is also of interest. This one concerns the number of points in the joint distribution of P_o and C. Large distributions require much more computer time to utilize, while small distributions may reduce resolution. Two cases are shown in Figure 7, a 25 point and a 121 point distribution. The two are both discrete approximations of a uniform distribution with the same expected values. As can be seen, there is very little difference between the two curves. Moreover, the 25 point case runs in one-fourth the computer time.

Certainly the number of cases run are not sufficient to draw general conclusions, but the sensitivity of the model does not appear to be too great, and the curves based on

Figure 5

GROWTH CURVE SENSITIVITY AFTER ONE TEST

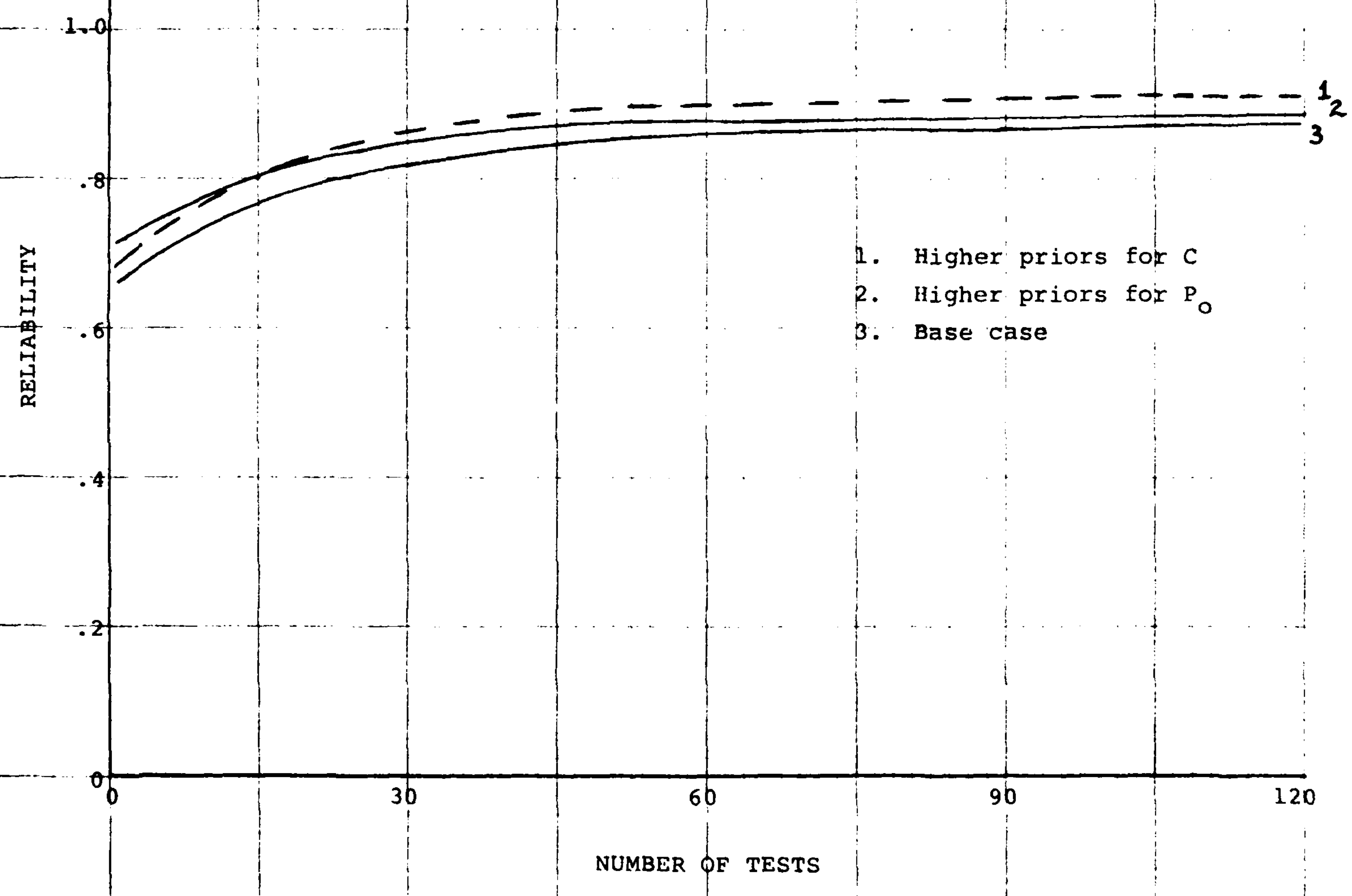


Figure 6

GROWTH CURVE SENSITIVITY AFTER 109 TESTS

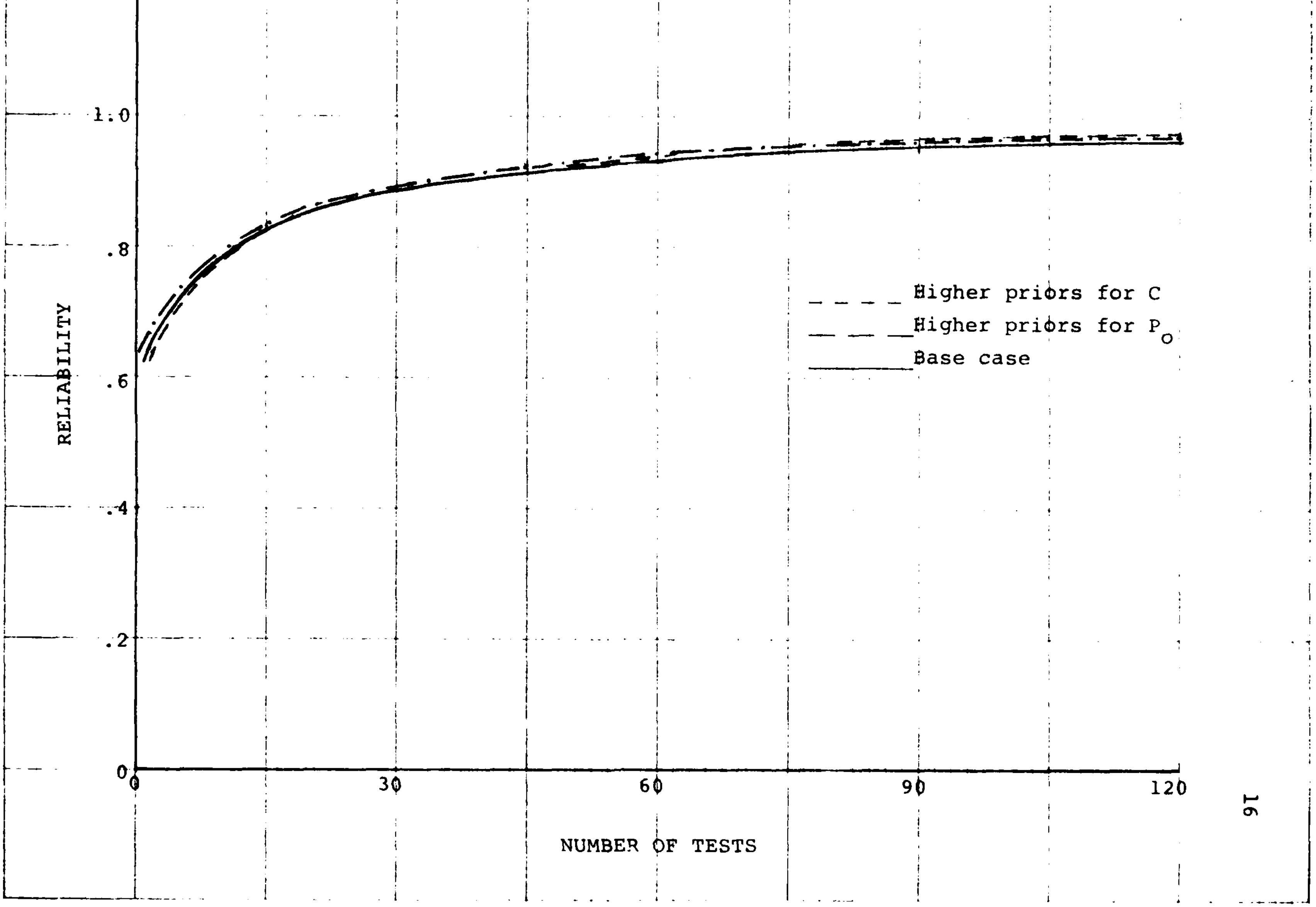
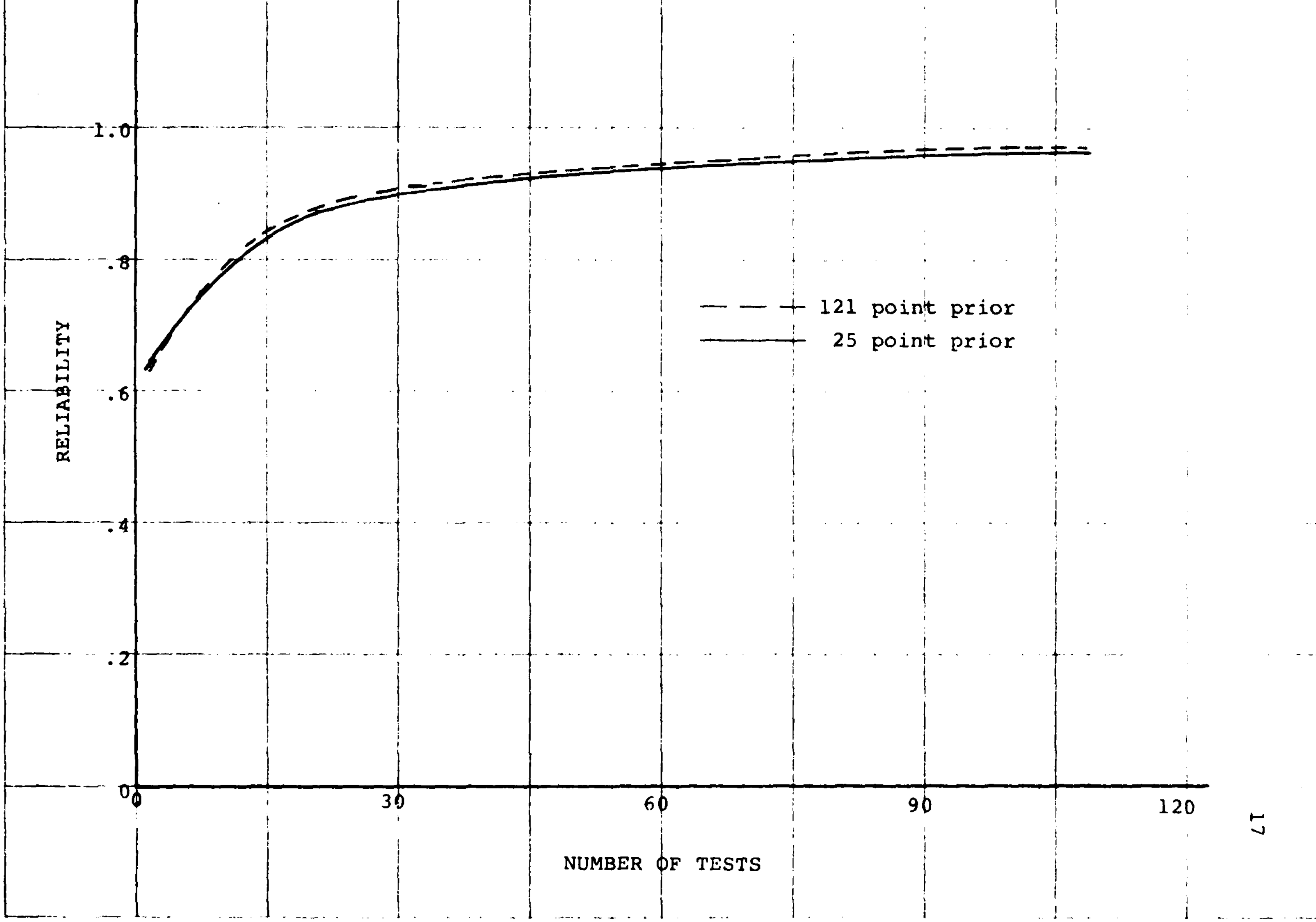


Figure 7

GROWTH CURVE SENSITIVITY TO NUMBER OF POINTS IN PRIOR



small numbers of tests are surely more meaningful than Duane curves based on similar numbers of tests.

Extension to MTBF

The model, as described thus far is designed for one-shot items which either succeed or fail. It is not applicable to programs where the system in question is tested until failure, the time-to-fail noted, and the design improved. However, it is suggestive of a model which is applicable to such programs.

Consider a repairable system whose time-to-fail is distributed exponentially. Let λ_i be the failure rate after i failures (and fixes). Let a sequence of tests (and fixes) be performed on the system with X_i being the time of the i-th failure. Also, let

$$Y_i = X_i - X_{i-1}.$$

Then the likelihood function for a sequence of N failures is

$$L = \lambda_o \cdot \lambda_1 \cdot \lambda_2 \cdots \lambda_{N-1} e^{-\lambda_o Y_1} e^{-\lambda_1 Y_2} \cdots e^{-\lambda_{N-1} Y_N},$$

and

$$\ell n(L) = \sum_{i=o}^{N-1} \ell n(\lambda_i) - \sum_{i=o}^{N-1} \lambda_i Y_{i+1}.$$

If some relation exists between successive λ_i, then the problem can be simplified. A useful example is

$$\lambda_{i+1} = \alpha \lambda_i$$

which implies

$$\lambda_i = \alpha^i \lambda_o.$$

If $\alpha > 1$, the system is degrading. If $\alpha < 1$, the system is improving. It follows that

$$\ell n(L) = N\,\ell n\,\lambda_o - \sum_{i=o}^{N-1} \alpha^i \lambda_o Y_{i+1} + \frac{N(N-1)}{2}\,\ell n\,\alpha$$

The maximum likelihood estimates of λ_o and α are found from:

$$\frac{\partial \ell n L}{\partial \lambda_o} = o = \frac{N}{\lambda_o} - \sum_{i=o}^{N-1} \alpha^i Y_{i+1}$$

and

$$\frac{\partial \ell n L}{\partial \alpha} = o = -\lambda_o \sum_{i=o}^{N-1} i\alpha^{i-1} Y_{i+1} + \frac{N(N-1)}{2}\,\frac{1}{\alpha}$$

Thus

$$\lambda_o = \frac{N}{\sum_{i=o}^{N-1} \alpha^i Y_{i+1}}$$

and α is a root of the equation

$$\sum_{i=o}^{N-1} \left(\frac{N-1}{2} - i\right) \alpha^i Y_{i+1} = o$$

A numerical method such as Newton-Rhapson can be used to find α in the latter equation which can then be used to solve for λ_o. This method has been applied to System #2 test data (using all 109 test results). The resulting maximum likelihood values for λ_o and α are

$$\lambda_o = .428567$$

$$\alpha = .798853$$

These parameters are related to the previous model as follows:

$$\lambda_o \doteq 1 - P_o$$

and

$$\alpha \doteq 1 - C.$$

However, MTBF testing is often for a specified total time T, and the expected MTBF after time T will be

$$\sum_{i=o}^{\infty} \frac{1}{\lambda_i} P_i(T) = \tau$$

where $P_i(T)$ is the probability that i events have occurred by time T. $P_i(T)$ is obtained from the solutions to the following system of equations:

$$\frac{dP_o}{dt} = -\lambda_o P_o$$

$$\frac{dP_i}{dt} = -\lambda_i P_i + \lambda_{i-1} P_{i-1}$$

Multiplying by $\frac{1}{\lambda_i}$ and summing, one obtains

$$\frac{d\tau}{dt} = -1 + \frac{1}{\alpha}$$

Hence

$$\tau = \frac{1}{\lambda_o} - \frac{\alpha-1}{\alpha} T$$

Conclusions and Recommendations

This study of some of the characteristics of the stochastic learning model for reliability growth prediction and assessment illustrates several characteristics of the model.

1. After many tests, the results compare favorably with existing methods.
2. For early assessment (after few tests), the

stochastic model is more applicable than existing methods.

3. Generating the priors is directly related to the pre-program planning process.
4. Probability of success monitoring is really a way of periodically reassessing the test program.
5. The model does not appear to be unusually sensitive to priors or size of the input distributions.
6. The model has a natural extension to repairable systems and mean time between failure calculation.
7. Programming and running on modern high-speed computers is easily accomplished.

These features appear to be sufficient to recommend the method. However, only three systems have been evaluated, and these were accomplished after the fact. Therefore, it is recommended that this method be used on several programs from pre-planning through assessment to determine its utility. This would not require that other methods be abandoned until or unless the suggested method prove more useful. At that point, a decision could be made on continuation or modification.

At the very foundation of this method is the continuing assessment of whether the program is progressing rapidly enough to meet its final goal. As a management tool, this should be invaluable. Moreover, the model is based on behavioral aspects as well as the complexity of the system. It is felt, though not yet substantiated, that decisions made will be more accurately modeled by the method described.

APPENDIX A

Scope of Work

1. General

This Scope of Work consists of engineering effort required to pursue the evaluation/development of a specific reliability growth model. The particular model of interest is a Bayesian learning model of the form $P_{K+1} = P_K + C(1 - P_K)$ where:

P_K = Probability of Kth Event

P_{K+1} = Probability of K + 1st Event

C = Constant Related to Learning Rate

2. Objective

In reliability growth planning/projection a model is needed that can be used to accurately predict the reliability growth of one-shot items such as missiles. The model shall have the capability to initially utilize estimates of the parameters of the model (Bayesian priors, etc.) and supplant these estimates with actual test data as it is generated. Existing models (Duane and others) are not totally satisfactory for projecting one-shot item reliability growth, especially in the early phases of a program and when data points (item test data) are few in number. The objective of this effort is to examine in detail a model of the type specified above and determine if it can be used/modified to satisfy this need.

3. Specific Tasks

a. Utilizing actual system test data provided by the MIRADCOM Product Assurance Directorate on 3-4 recently developed missile systems, determine if the model would have achieved a good fit of the data. Compare the fit achieved on each system (with the model) with the AMSAA Model (Duane Model) fit of the same data.

b. Develop complete methodology for applying the model to reliability growth projections for systems just entering the development cycle.

c. Examine the model to determine its relative sensitivity to estimates of the model parameters. Determine how quickly errors in initial estimates of the parameters of the model are reduced by progressive use of actual data as it is generated.

d. After accomplishment of tasks a, b, and c above, provide recommendations on appropriateness of the model for use by MIRADCOM for reliability growth planning including any changes/modifications deemed appropriate to give the model more utility. This should include consideration of modifying the model so that it can be used for mean-time-between-failure data as well as one-shot data.

4. Reporting Requirements

a. Oral reports as requested.

b. A final report (reproducible master plus one copy) summarizing the work performed and conclusions derived under each task shall be submitted to the COTR within 30 days after completion of services.

APPENDIX B

Growth Curve Data

SYSTEM #1

Stochastic Model

Test No.	Reliability	Test No.	Reliability
5	.453871	189	.820646
9	.477323	193	.823222
13	.498814	197	.825725
17	.518584	201	.828158
21	.536837	205	.830524
25	.553741	209	.832826
29	.569444	213	.835066
33	.58407	217	.837247
37	.597728	221	.839371
41	.610511	225	.84144
45	.622501	229	.843457
49	.633771	233	.845423
53	.644383	237	.84734
57	.654395	241	.84921
61	.663855	245	.851035
65	.672809	249	.852816
69	.681297	253	.854555
73	.689353	257	.856254
77	.69701	261	.857913
81	.704298	265	.859535
85	.711242	269	.861119
89	.717866	273	.862669
93	.724192	277	.864184
97	.73024	281	.865666
101	.736027		
105	.741571		
109	.746886		
113	.751986		
117	.756884		
121	.761592		
125	.76612		
129	.770481		
133	.774683		
137	.778733		
141	.78264		
145	.786412		
149	.790055		
153	.793576		
157	.796981		
161	.800275		
165	.803464		
169	.806553		
173	.809546		
177	.812448		
181	.815263		
185	.817994		

Duane Curve $(1-\lambda\beta N^{\beta-1})$

$\beta = .643469 \qquad \lambda = 1.78815$

Test No.	Reliability	Test No.	Reliability
5	.351773	189	.822457
9	.474328	193	.823778
13	.53892	197	.825062
17	.580976	201	.826312
21	.611385	205	.827528
25	.634807	209	.828712
29	.653629	213	.829866
33	.669224	217	.83099
37	.682445	221	.832087
41	.693857	225	.833158
45	.703851	229	.834203
49	.712708	233	.835223
53	.720634	237	.83622
57	.727788	241	.837195
61	.734291	245	.838147
65	.74024	249	.839079
69	.745713	253	.839991
73	.750771	257	.840883
77	.755466	261	.841757
81	.759842	265	.842613
85	.763934	269	.843451
89	.767773	273	.844273
93	.771384	277	.845078
97	.774791	281	.845868
101	.778012		
105	.781065		
109	.783964		
113	.786722		
117	.789351		
121	.791861		
125	.79426		
129	.796558		
133	.798761		
137	.800876		
141	.802908		
145	.804864		
149	.806748		
153	.808565		
157	.810318		
161	.812012		
165	.81365		
169	.815234		
173	.816769		
177	.818256		
181	.819699		
185	.821098		

SYSTEM #2

Stochastic Model				Duane Curve $(1-\lambda\beta N^{\beta-1})$ $\beta = .60512$ $\lambda = .643436$			
Test No.	Reliability	Test No.	Reliability	Test No.	Reliability	Test No.	Reliability
2	.648134	50	.924783	2	.703874	50	.916927
3	.676819	51	.925932	3	.747686	51	.917575
4	.700693	52	.927046	4	.774781	52	.918204
5	.720944	53	.928125	5	.793777	53	.918817
6	.738385	54	.929172	6	.808102	54	.919414
7	.753594	55	.930188	7	.819435	55	.919996
8	.766993	56	.931174	8	.828709	56	.920563
9	.778903	57	.932132	9	.836493	57	.921116
10	.789569	58	.933063	10	.843156	58	.921656
11	.799185	59	.933968	11	.84895	59	.922183
12	.807904	60	.934848	12	.854051	60	.922698
13	.81585	61	.935705	13	.858592	61	.923201
14	.823127	62	.936539	14	.862671	62	.923692
15	.829817	63	.93735	15	.866361	63	.924173
16	.835992	64	.938141	16	.869724	64	.924643
17	.841711	65	.938911	17	.872806	65	.925103
18	.847023	66	.939662	18	.875645	66	.925553
19	.851973	67	.940394	19	.878271	67	.925994
20	.856596	68	.941109	20	.880712	68	.926426
21	.860926	69	.941806	21	.882988	69	.926849
22	.864989	70	.942486	22	.885118	70	.927263
23	.868811	71	.94315	23	.887117	71	.927669
24	.872413	72	.943798	24	.888998	72	.928068
25	.875814	73	.944432	25	.890773	73	.928459
26	.879031	74	.945051	26	.892452	74	.928842
27	.882078	75	.945656	27	.894043	75	.929218
28	.884969	76	.946247	28	.895554	76	.929587
29	.887716	77	.946826	29	.896991	77	.92995
30	.89033	78	.947391	30	.898361	78	.930306
31	.89282	79	.947945	31	.899668	79	.930656
32	.895195	80	.948487	32	.900918	80	.930999
33	.897463	81	.949017	33	.902115	81	.931337
34	.899631	82	.949537	34	.903262	82	.931669
35	.901707	83	.950045	35	.904363	83	.931995
36	.903695	84	.950543	36	.905421	84	.932316
37	.905601	85	.951032	37	.906439	85	.932631
38	.907431	86	.95151	38	.907419	86	.932942
39	.909189	87	.951979	39	.908364	87	.933247
40	.910879	88	.952439	40	.909275	88	.933548
41	.912505	89	.952889	41	.910156	89	.933844
42	.914071	90	.953331	42	.911007	90	.934135
43	.91558	91	.953765	43	.91183	91	.934422
44	.917035	92	.954191	44	.912626	92	.934704
45	.918439	93	.954608	45	.913398	93	.934982
46	.919795	94	.955018	46	.914147	94	.935256
47	.921105	95	.955421	47	.914873	95	.935526
48	.922372	96	.955816	48	.915577	96	.935792
49	.923597	97	.956204	49	.916262	97	.936054

SYSTEM #2 (Con't.)

Stochastic Model				Duane Curve $(1-\lambda\beta N^{\beta-1})$ β = .60512 λ = .643436			
Test No.	Reliability	Test No.	Reliability	Test No.	Reliability	Test No.	Reliability
98	.956585	104	.958736	98	.936313	104	.93779
99	.956959	105	.959074	99	.936568	105	.938025
100	.957327	106	.959406	100	.936819	106	.938256
101	.957688	107	.959732	101	[illegible]7067	107	.938485
102	.958044	108	.960054	102	[illegible]7311	108	.93871
103	.958393	109	.96037	103	[illegible]7552	109	.938933

SYSTEM #3

Stochastic Model

Test No.	Reliability	Test No.	Reliability
2	.472249	50	.897904
3	.517579	51	.89956
4	.55523	52	.901162
5	.58709	53	.902715
6	.614456	54	.904218
7	.638253	55	.905676
8	.659159	56	.90709
9	.677689	57	.908462
10	.694238	58	.909794
11	.709115	59	.911087
12	.722569	60	.912344
13	.734799	61	.913565
14	.745969	62	.914753
15	.756213	63	.915909
16	.765644	64	.917033
17	.774358	65	.918128
18	.782433	66	.919194
19	.78994	67	.920232
20	.796937	68	.921244
21	.803475	69	.922231
22	.809599	70	.923193
23	.815347	71	.924131
24	.820752	72	.925047
25	.825846	73	.925941
26	.830655	74	.926814
27	.835202	75	.927666
28	.839508	76	.928498
29	.843592	77	.929312
30	.847472	78	.930107
31	.851161	79	.930885
32	.854675	80	.931645
33	.858024	81	.932389
34	.861222	82	.933117
35	.864277	83	.933829
36	.867199	84	.934526
37	.869997	85	.935208
38	.872679	86	.935877
39	.875252	87	.936531
40	.877721	88	.937173
41	.880094	89	.937801
42	.882376	90	.938418
43	.884573	91	.939022
44	.886688	92	.939614
45	.888726	93	.940195
46	.890692	94	.940764
47	.89259	95	.941323
48	.894422	96	.941872
49	.896192	97	.94241

Duane Curve

$\beta = .501774 \quad \lambda = 1.43041$

Test No.	Reliability	Test No.	Reliability
2	.491855	50	.897789
3	.584803	51	.898792
4	.640245	52	.899767
5	.678098	53	.900714
6	.70605	54	.901634
7	.727781	55	.902529
8	.745302	56	.9034
9	.759818	57	.904248
10	.772101	58	.905074
11	.78267	59	.90588
12	.79189	60	.906664
13	.800026	61	.90743
14	.807275	62	.908177
15	.813787	63	.908906
16	.81968	64	.909618
17	.825045	65	.910313
18	.829957	66	.910993
19	.834476	67	.911657
20	.838653	68	.912307
21	.842528	69	.912942
22	.846135	70	.913564
23	.849506	71	.914173
24	.852663	72	.914769
25	.855629	73	.915353
26	.858423	74	.915925
27	.86106	75	.916485
28	.863555	76	.917034
29	.86592	77	.917573
30	.868166	78	.918101
31	.870302	79	.918619
32	.872337	80	.919128
33	.87428	81	.919627
34	.876136	82	.920117
35	.877912	83	.920598
36	.879613	84	.92107
37	.881246	85	.921534
38	.882813	86	.92199
39	.88432	87	.922438
40	.88577	88	.922878
41	.887167	89	.923311
42	.888513	90	.923737
43	.889813	91	.924156
44	.891067	92	.924567
45	.89228	93	.924973
46	.893453	94	.925371
47	.894589	95	.925764
48	.895689	96	.92615
49	.896755	97	.92653

SYSTEM #3 (Con't.)

Stochastic Model				Duane Curve β = .501774 λ = 1.43041			
Test No.	Reliability	Test No.	Reliability	Test No.	Reliability	Test No.	Reliability
98	.942938	111	.949017	98	.926905	111	.931303
99	.943457	112	.949432	99	.927274	112	.93161
100	.943966	113	.949839	100	.927637	113	.931912
101	.944467	114	.95024	101	.927995	114	.93221
102	.944958	115	.950635	102	.928347	115	.932504
103	.945441	116	.951024	103	.928695	116	.932795
104	.945915	117	.951406	104	.929037	117	.933082
105	.946381	118	.951783	105	.929375	118	.933365
106	.946839	119	.952153	106	.929708	119	.933644
107	.94729	120	.952518	107	.930036	120	.933921
108	.947733	121	.952878	108	.930359	121	.934193
109	.948168	122	.953232	109	.930678	122	.934462
110	.948596	123	.953581	110	.930993	123	.934728

APPENDIX C

Priors For Sensitivity Analyses

Base Case

As given for Systems #2 and #3 on page 5.

Higher Growth (c)

C \ P_o	.25	.40	.55	.70	.85
0	.02	.02	.02	.02	.02
.06	.03	.03	.03	.03	.03
.12	.04	.04	.04	.04	.04
.18	.05	.05	.05	.05	.05
.24	.06	.06	.06	.06	.06

Higher P_o

C \ P_o	.25	.40	.55	.70	.85
0	.02	.03	.04	.05	.06
.06	.02	.03	.04	.05	.06
.12	.02	.03	.04	.05	.06
.18	.02	.03	.04	.05	.06
.24	.02	.03	.04	.05	.06

REFERENCE

[1] Bush, R. and Mosteller, F., "A Stochastic Model with Applications to Learning", Annals of Math. Stat., 1953, Vol. 24, 559-585.